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Instability of conducting fluid cylinder due to axial current

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With 1 figure in the text

ABSTRACT

The stability of an incompressible, inviscid, electrically conducting fluid cylinder in the presence of axial current is considered taking into account the effect of surface tension. The effect of gravity is taken into account approximately. The results are found to be in reasonable agreement with experiment.

1. Introduction

In recent years the theory of stability of constricted gas discharges was studied in great detail in view of its importance in the problem of controlled production of thermonuclear energy. Kruskal and Schwarzchild [1], Taylor [2, 3], among others, showed that a pinched gas discharge is highly unstable for "sausage" and "kink" deformations. Experimental evidence [4] confirms that theoretical results are qualitatively correct.

In view of the importance of the stability problem, one would like to have quantitative comparison between theory and experiment under suitable conditions. With this end in view a liquid scale model of pinched gas discharge has been constructed by Dattner, Lehnert and Lundquist [5]. In their experiment a mercury jet of 14 cm length and 0.2 cm radius, falling vertically from a circular hole in a tank, plays the role of a pinched gas discharge. It is observed that when an axial current is passed through the mercury jet without any superimposed axial magnetic field, the jet develops a sausage instability while in the presence of axial magnetic field and axial current it develops corkscrew instability. This experiment is not only free from the difficulties of pinch discharges, but also has the merit that it allows quantitative comparison with theory while bringing forth the qualitative information about instability of pinched gas discharges. It is the aim of this paper to develop a theory for axi-symmetric perturbation in the presence of an axial current without an axial magnetic field and compare the results of the theory with the above-mentioned experiment. The effect of an axial external magnetic field will be considered in a later paper.

2. Basic assumptions and equations

Let us consider (as a working model) an infinitely long fluid cylinder of circular cross-section of radius R_0 surrounded by vacuum. We assume that the fluid is incompressible and inviscid and has a finite electrical conductivity. A current I is assumed to be flowing along the axis of the cylinder. In the following analysis we shall first ignore the effect of gravity, but discuss this effect at the end.

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In C.G.S. electromagnetic units (ignoring the effect of displacement current) the basic equations are

$$\varrho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{H}, \tag{2.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}, \tag{2.3}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2.4}$$

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{H}), \tag{2.5}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{2.6}$$

and

$$I = \text{constant}. \tag{2.7}$$

The symbols in the eqns. (2.1) to (2.6) have their usual meaning [6]. In the last eqn. (2.7) I = total current passing along the fluid cylinder. All the eqns. (2.1) to (2.6) are the basic equations of any problem in hydromagnetics of an incompressible fluid of finite electrical conductivity. Eqn. (2.7) is a constraint on the system in the present case denoting that the source of current is outside fluid cylinder and supplies a current of constant value.

Equilibrium

The equilibrium state of the fluid cylinder can be described by

$$0 = -\nabla p_e + \mathbf{j}_e \times \mathbf{H}_e, \tag{2.8}$$

and eqns. (2.3) and (2.7). If j_0 is the equilibrium current density along the axis of the fluid cylinder which we assume coincides with z-axis of circular cylindrical coordinates, we have

$$j_0 = \frac{I}{\pi R_0^2}. \tag{2.9}$$

One can show that the pressure, p_i, in the interior of the fluid cylinder at a distance $r \leqslant R_0$ from the axis is given by

$$p_e \equiv p_i = \frac{T}{R_0} + \pi j_0^2 (R_0^2 - r^2), \tag{2.10}$$

where T = surface tension of the fluid (mercury in our problem).

3. Criterion of instability of the equilibrium for axi-symmetric perturbation and the method of solution

In order to determine whether the equilibrium state given by eqns. (2.9) and (2.10) is stable or not, we assume that the fluid cylinder is subjected to an axi-symmetric perturbation which deforms the surface of the cylinder into a shape given (in circular cylindrical co-ordinates with z-axis coinciding with the axis of the cylinder) by

$$r = R + a\cos kz, \tag{3.1}$$

where a is a function of time subject to the condition

$$a \ll R, \tag{3.2}$$

so that we can neglect terms proportional to a^2. Due to the fact the fluid is assumed to be incompressible we find that

$$R_0^2 = R^2 + \frac{a^2}{2}, \tag{3.3}$$

so that one can ignore the difference between R_0 and R. If we assume further that a is periodic in time so that

$$a = a_0 \cos \omega t, \tag{3.4}$$

the problem of investigating the stability of the fluid cylinder will be shifted to finding the values of ω. If ω is found to have an imaginary value for any real value of k we conclude that the fluid cylinder is unstable for perturbations that deform its surface into a shape given by eqn. (3.1). The values of ω can be found from eqn. (2.1) if we know the distribution of pressure, p, as well as current density and magnetic field in the perturbed condition. These quantities will be calculated in the following sections.

4. Distribution of perturbed current density and magnetic field

The fundamental quantity in the calculation of ω is the perturbed current density distribution. We shall calculate this quantity in the "magnetostatic approximation". This approximation is good when the electrical conductivity of the fluid is so small that the characteristic time for diffusion of magnetic lines of force is smaller than the characteristic time for fluid motions [7]. A more direct way of looking at this approximation is as follows.

If we combine eqns. (2.4) and (2.5) we obtain

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \left[\frac{\mathbf{j}}{\sigma} - \mathbf{v} \times \mathbf{H}\right]. \tag{4.1}$$

If the electrical conductivity σ of the fluid is infinitely large eqn. (4.1) leads to the well-known equation of induction for frozen magnetic lines of force:

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}). \tag{4.2}$$

If, on the contrary, the electrical conductivity of the fluid is vanishingly small we find that

$$\nabla \times \mathbf{j} = 0, \tag{4.3}$$

so that the current density can be derived as a gradient of scalar function, say, ψ:

$$\mathbf{j} = \nabla \psi. \tag{4.4}$$

As far as magnetohydrodynamic problems are concerned mercury can be considered as a poor electrical conductor and therefore we assume that eqn. (4.4) is valid. By virtue of eqn. (2.3), ψ satisfies the Laplace equation:

$$\nabla^2 \psi = 0. \tag{4.5}$$

In order to determine uniquely the perturbed current density distribution, we impose the constraint given by eqn. (2.7) in an equivalent form. We demand that there is no current flowing along the normal to the perturbed boundary, or

$$(\mathbf{j} \times \mathbf{n})_{r = R + a \cos kz} = 0, \tag{4.6}$$

where $\mathbf{n}$ is the unit normal to the surface of the deformed cylinder. One can show that

$$\mathbf{n} = (1, 0, 0) + (0, 0, ak) \sin kz. \tag{4.7}$$

We remark here that eqns. (4.3) to (4.6) are formally the same as those considered by Chandrasekhar and Fermi [8] in connection with stability of an infinitely conducting incompressible fluid cylinder with an axial magnetic field frozen in the medium. Following their method one can show that the perturbed current density in the present problem is given by

$$\left.\begin{aligned} &\mathbf{j} = (j_r, 0, j_z), \\ \text{where} \quad &j_r = -ak j_0 \frac{I_1(kr)}{I_1(kR)} \sin kz, \\ \text{and} \quad &j_z = j_0 - ak j_0 \frac{I_0(kr)}{I_1(kR)} \cos kz, \end{aligned}\right\} \tag{4.8}$$

where $I_n(kr)$ is the modified Bessel function of order n. Using eqns. (2.3) and (4.8) we find that the perturbed magnetic field distribution is given by

$$\mathbf{H} = (0, H_\theta, 0), \tag{4.9}$$

where

$$H_\theta = \begin{cases} 2\pi j_0 r - 4\pi a j_0 \dfrac{I_1(kr)}{I_1(kR)} \cos kz, & \text{for} \quad r \leqslant R + a \cos kz, \\ 2\pi j_0 R^2/r, & \text{for} \quad r \geqslant R + a \cos kz. \end{cases} \tag{4.10}$$

5. Distribution of pressure inside the deformed cylinder

The knowledge of the distribution of current density inside the deformed cylinder enables us to calculate the distribution of pressure. If we take the divergence of both sides of eqn. (2.1) and use eqns. (2.2), (2.3), (4.4) and (4.8) we find, retaining terms up to first order, that

$$\nabla^2 p = -4\pi j_0^2 + 8\pi a k j_0^2 \frac{I_0(kr)}{I_1(kR)} \cos kz. \tag{5.1}$$

In order to solve eqn. (5.1) let us write

$$p = p_0(r) + a\, p_1(r) \cos kz. \tag{5.2}$$

If we substitute this expression for p in eqn. (5.1) and separate terms of corresponding orders we obtain two inhomogeneous differential equations which can be solved by standard methods. We only give the result which shows that

$$\left.\begin{aligned} & p_0(r) = A - \pi j_0^2 r^2, \\ \text{and}\qquad & p_1(r) = B\, I_0(kr) + 4\pi j_0^2 r \frac{I_0(kr)}{I_1(kR)} \end{aligned}\right\}, \tag{5.3}$$

where A and B are arbitrary constants to be determined by boundary conditions.

To determine the constants A and B we note that the pressure, p_b, at a point just inside the deformed boundary depends only on the surface tension and curvature of the surface since we assumed that the pressure outside the fluid cylinder is zero. One can show that [9]

$$p_b = \frac{T}{R} + \frac{aT}{R^2}(k^2R^2 - 1)\cos kz. \tag{5.4}$$

The boundary condition we impose on the solution of eqn. (5.1) is that the right-hand side of eqn. (5.2) evaluated at $r = R + a \cos kz$ equals the right-hand side of eqn. (5.4). We then find that

$$\left.\begin{aligned} & A = \frac{T}{R} + \pi j_0^2 R^2, \\ \text{and}\qquad & B = \frac{1}{I_0(kR)}\left[\frac{T}{R^2}(k^2R^2 - 1) - 2\pi j_0^2 R\right] \end{aligned}\right\}. \tag{5.5}$$

Therefore, eqns. (5.2), (5.3) and (5.5) define uniquely (apart from a trivial additive constant) the distribution of pressure inside the deformed cylinder in the range $0 \leqslant r \leqslant R + a \cos kz$ up to terms of order a.

6. Dispersion relation

The calculation of ω is relatively simple matter since we obtained all the quantities on the right-hand side of eqn. (2.1). If we use eqn. (3.4) we can write (neglecting terms of higher orders) the radial part of eqn. (2.1) as

$$-\varrho\,\omega^2 a \cos kz = \left\{-\frac{\partial p}{\partial r} + (\mathbf{j}\times\mathbf{H})_r\right\}_b, \tag{6.1}$$

where the suffix to the brace on the right-hand side of eqn. (6.1) indicates that the quantities inside that brace must be evaluated at the deformed boundary. Using eqns. (4.8), (4.10), (5.2), (5.3) and (5.5) we find that eqn. (6.1) gives the required dispersion relation:

$$\Omega^2 = x(x^2-1)\frac{I_1(x)}{I_0(x)} - 4N\left\{1+\frac{x}{2}\left(\frac{I_1(x)}{I_0(x)}-\frac{I_0(x)}{I_1(x)}\right)\right\}, \tag{6.2}$$

where $$x \equiv kR,$$

$$N \equiv I^2/\pi R T, \tag{6.3}$$

and $$\Omega^2 \equiv \varrho\,\omega^2 R^3/T. \tag{6.4}$$

We can draw the following conclusions from eqn. (6.2).

(*i*) Ω^2 is a real number for real values of k which means that there are no states of overstability, a result proved by Bernstein *et al.* [10] in the case of infinitely conducting plasma.

(*ii*) By taking the asymptotic values of the Bessel functions one can show that

$$\underset{x\to 0}{Lt}\,\Omega^2 \to -x^2(1+N)/2,$$

and $$\underset{x\to\infty}{Lt}\,\Omega^2 \to +\infty,$$

for all finite values of N. Since the modified Bessel functions are monotonically changing functions of the argument we can conclude from the above asymptotic limits that Ω^2 has one more zero, apart from the one it has at the origin, at some value of x which we may denote by x_c. In the range $0 < x < x_c$, we find that $\Omega^2 < 0$ and has a minimum at some value of x which we denote by x_m. Therefore in the range $0 < x < x_c$ the values of ω are pure imaginary and hence we conclude that for disturbances that impress on the cylinder wavelengths corresponding to the range $0 < x < x_c$, the fluid cylinder is unstable. For values of wavelength corresponding to $x = x_m$ the instability

Table 1. Variation of $x_c = 2\pi R/\lambda_c$ and $x_m = 2\pi R/\lambda_m$ with $N = I^2/\pi RT$.

The values of x_c and x_m are accurate up to ± 0.02 due to graphical method of solution adapted for obtaining them.

N	x_c	x_m	x_c/x_m
0	1.00	0.695	1.44
1	1.37	0.94	1.46
2	1.62	1.10	1.47
3	1.83	1.24	1.48
4	2.01	1.34	1.50
5	2.17	1.44	1.51

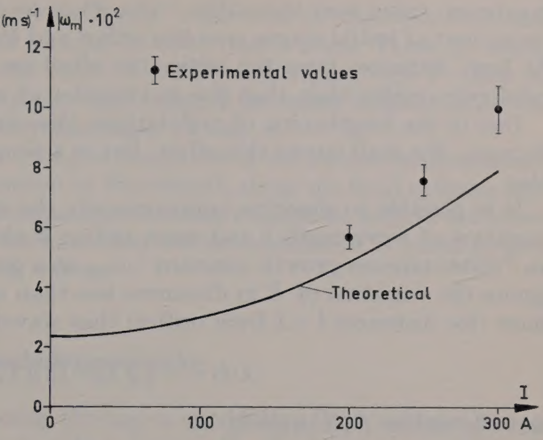

Fig. 1. Variation of growth constant $|\omega_m|(ms)^{-1}$ with axial current I (Amp) for mercury jet of 0.2 cm radius; $T = 487$ dyne cm^{-1}; $\varrho = 13.6$ g cm^{-3}. Axi-symmetric perturbation.

grows at a maximum rate. We further note that the term inside the brace of eqn. (6.2) is always positive, thereby contributing a negative quantity to Ω^2 since $N > 0$. This shows the unstabilising effect of current flowing along the axis of the fluid cylinder.

For a given value of N the values of x_c and x_m are obtained graphically and the results are shown in the Table 1.

Corresponding to the value of $x = x_m$ the value of $|\omega_m|$ is calculated as a function of current I (amp) for mercury jet of 0.2 cm radius ($T = 487$ dyne cm^{-1} and $\varrho = 13.6$ g cm^{-3}) and the results are shown in Fig. 1. In the same figure the values of $|\omega|$ measured from the film taken by Dattner [5] are also plotted. The experimental values are consistently larger than the theoretical values.

There are mainly three possible reasons for the discrepancy between the experimental and theoretical results.

(1) The actual surface tension of mercury can be smaller than the value used in the theoretical calculations on account of surface impurities.

(2) The effect of viscous forces is neglected in theory. This is a good approximation in the present problem since the effect of viscous forces is several orders of magnitude smaller than electromagnetic forces in the case of mercury [11].

(3) The jet is falling vertically under gravity. This is expected to appreciably affect the stability and therefore will be discussed in some detail in the next section.

7. Effect of gravity

It has been assumed in the theory developed so far that the fluid is unaffected by gravity. If the fluid is falling vertically downwards due to gravity we have to take into account (1) the decrease of cross-section of the fluid cylinder and (2) the lengthening of undulations that develop due to instability as the velocity increases. As the jet of fluid emerges from the orifice the disturbance that impresses upon it undulations of length approximately equal to 0.8 times the circumference of orifice (when $I = 300$ A) are most unstable. But as the jet moves faster the parameter $x = 2\pi R/\lambda$ decreases due to effects (1) and (2) and therefore the value of ω changes from its maximum value near the orifice.[1] The effect on ω due to variation of radius is more prominent at initial stages near the orifice and for smaller initial velocities of the jet. At large distances from the orifice the effect on ω due to the variation of radius is relatively smaller than that due to variation of wavelength.

Due to the lengthening of undulations, the curvature of the surface of the jet also changes. We shall ignore this effect, but in a complete theory one should include this also.

It is possible to describe approximately the effect of gravity by considering the variation of wavelength λ and mean radius R along the axis of the jet, and defining an "instantaneous growth constant" ω_{inst} at a given distance l from the orifice. If we ignore the variation of R in distances less than or equal to one wavelength, one can show (for distances $l > \lambda$ from orifice) that wavelength $\lambda(l)$ is given by

$$\lambda(l) = -\tfrac{1}{2} g T_0^2 + [2 g T_0^2 (h + l)]^{\frac{1}{2}}, \tag{7.1}$$

and the radius $R(l)$ is given by

$$R(l) = R_0 \Big/ \left[1 + \frac{l}{h}\right]^{\frac{1}{4}}. \tag{7.2}$$

In eqn. (7.1) T_0 is defined as the time interval in which the fluid emerging from the orifice moves a distance equal to λ_m or

$$\lambda_m \equiv T_0 (2 g h)^{\frac{1}{2}} + \tfrac{1}{2} g T_0^2, \tag{7.3}$$

where h = the height of the fluid above orifice and g = the acceleration due to gravity. T_0 is calculated from the theoretical value of λ_m defined in the gravity free case. In the experiment of Dattner [5] the value of h is about 3 cm. Taking $h = 3$ cm the value of ω_{inst} for various values of l is calculated for $I = 300$ A and the results are shown in Table 2. The large initial value of ω_{inst} is due to the rapid change of radius of the jet as it emerges from the orifice. The decrease of ω_{inst} at larger distances from orifice is due to the increase of wavelength which dominates the effects due to variation in radius as may be seen from eqns. (7.1) and (7.2).

Calculations show that the above results are qualitatively the same when the current is zero. This should be contrasted with the conclusion of Rayleigh [12], namely, that "the change of conditions during fall (under gravity) has a protective influence". This is true only at great distances from the orifice but at closer distances the changing conditions have an unstabilising effect.

Table 2. Variation of "instantaneous growth constant", ω_{inst}, due to gravity along the mercury jet.

l is the distance from the orifice of radius 0.2 cm, the axial current is 300 A.

l (cm)	2	4	6	8	10	12
$\omega_{\text{inst}} \times 10^2\ (ms)^{-1}$	9.5	9.4	9.2	8.9	8.7	8.4

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[1] In this section we shall write ω in place of $|\omega|$.

It is interesting to note that the "instantaneous" value of ω at l between 6 cm and 12 cm varies only by about 8 % with a mean ≈ 0.088 (ms)$^{-1}$ (refer Table 2) which should be compared with experimental value of 0.099 ± 0.010 (ms)$^{-1}$ for $I = 300$ A. (Fig. 1). We remark here that the experimental value is the mean result of observations made in the lower half of the jet (total length 14 cm).

Finally, we point out that the value of x_m at the orifice of the jet can be determined experimentally by observing the variation of wavelength along the fluid cylinder and extrapolating to the orifice. As the deformation becomes observable only in the lower half of the jet the extrapolation is a bit uncertain but nevertheless it is found to give reasonable agreement with theoretical value.

8. Concluding remarks

We have shown that a current passing through a conducting fluid cylinder has an unstabilising effect. In the absence of surface tension the fluid cylinder will break into pieces of very small length which should be compared with the instability of pinch discharge. The force due to surface tension has a stabilising effect at small wavelengths. When the fluid is falling under gravity, neglecting the effect of the variation of curvature of the deformed surface, the effect of variation of wavelength and radius can be approximately described by eqns. (7.1) and (7.2). Viscosity has been neglected but is expected not to change appreciably the above results in the case of mercury jet.

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